INFINITY ALGEBRAS AND THE HOMOLOGY OF GRAPH COMPLEXES

MICHAEL PENKAVA

Abstract. An L_∞ algebra is a generalization of a Lie algebra [10, 9, 17]. Given an L_∞ algebra with an invariant inner product, we construct a cycle in the homology of the complex of metric ordinary graphs. Since the cyclic cohomology of a Lie algebra determines infinitesimal deformations of the algebra into an L_∞ algebra, this construction shows that a cyclic cocycle of a Lie algebra determines a cycle in the homology of the graph complex. This result was suggested by a remark by M. Kontsevich in [7] that every finite dimensional Lie algebra with an invariant inner product determines a cycle in the graph complex. In a joint paper with A. Schwarz [14], we proved that an A_∞ algebra with an invariant inner product determines a cycle in the homology of the complex of metric ribbon graphs. In this article, a simpler proof of this fact is given. Both constructions are based on the ideas presented in [13].

1. Introduction

In [13], a definition of the cohomology of an L_∞ algebra was given, and it was shown that the cohomology of a Lie algebra classifies the (infinitesimal) deformations of the Lie algebra into an L_∞ algebra. The cyclic cohomology of a Lie algebra was defined, and it was shown that the cyclic cohomology of a Lie algebra with an invariant inner product classifies the (infinitesimal) deformations of the Lie algebra into an L_∞ algebra preserving the invariant inner product.

In a previous joint article with Albert Schwarz [14], similar results were obtained about the cohomology and cyclic cohomology of an associative algebra classifying the deformations of the algebra into an A_∞ algebra. In that article, we defined Hochschild cohomology and cyclic cohomology of an A_∞ algebra, and also applied these considerations to construct cycles in the homology of the complex of metric ribbon graphs, which is closely associated to the homology of the moduli spaces of Riemann surfaces with fixed genus and number of marked points.

The cohomology of A_∞ algebras was studied in more detail in [13], from the point of view of the coalgebra structure on the tensor coalgebra of a vector space, and the coalgebra structure on its parity reversion (we assume the space in question has a $\mathbb{Z}_2$-grading). A bracket structure, the Gerstenhaber bracket, was defined on the tensor algebra. This bracket arises from the bracket of coderivations on the

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tensor coalgebra of the parity reversion. The differential in the cohomology of an A_∞ algebra is given by the Gerstenhaber bracket with the cochain determined by the A_∞ structure, much as the differential of cochains of an associative algebra is given by the Gerstenhaber bracket with the cochain given by the associative multiplication, as was shown in [2].

When an A_∞ algebra possesses a (graded symmetric) inner product, there is a notion of a cyclic cochain. The property of cyclicity is preserved by the Gerstenhaber bracket, and the differential is given by the bracket when the inner product is invariant. One can also define a notion of cyclicity in the tensor coalgebra of the parity reversion, by means of the isomorphism that exists between the two structures. We will give this definition here. Another important fact is that symmetric inner products on a vector space give rise to graded antisymmetric inner products on its parity reversion.

The cohomology of an L_∞ algebra is given by considering the exterior coalgebra of a vector space. The parity reversion of the exterior coalgebra is the symmetric coalgebra of the parity reversion. There is a bracket defined on the exterior coalgebra which arises from the bracket of coderivations of the symmetric coalgebra of the parity reversion, and the differential in the cohomology of an L_∞ algebra is given by the bracket in much the same manner as in the A_∞ case.

The notion of a cyclic cochain for an L_∞ algebra with an invariant graded symmetric inner product yields a theory of cyclic cohomology of L_∞ algebras, where again, the differential is given by a bracket. We will need to consider here how the defining relations for an L_∞ algebra are expressed in the corresponding theory on the symmetric coalgebra of the parity reversion, which will possess a graded antisymmetric inner product.

In [7], an orientation on the complex of metric ordinary graphs was given. We will need an equivalent formulation of this orientation. Similarly, in [14] we gave a definition (following Kontsevich) of the orientation on the complex of metric ribbon graphs; we shall also need to reformulate this definition.

We will explain the graph complexes and the definition of homology in some detail, and then formulate the definitions of the cycles we are interested in. The results in the next two sections are really just a reformulation of results which were proved in [14, 13].

2. A_∞ ALGEBRAS

In what follows we shall assume that V is a finite dimensional $\mathbb{Z}_2$-graded vector space over a field $\mathbf{k}$ of characteristic zero. The $\mathbb{Z}_2$-grading means that V has a preferred decomposition $V = V_0 \bigoplus V_1$. An element in V_i has even parity, when $i = 0$, and odd parity when $i = 1$. The parity reversion ΠV is the same space, but with reversed parity; *i.e.*, $(\Pi V)_0 = V_1$ and $(\Pi V)_1 = V)0$. The tensor coalgebra $T(V)$ is defined by $T(V) = \bigoplus_{n=1}^\infty V^n$, where V^n is the n-th tensor power of V. An A_∞ algebra structure on V is given by an element (cochain) $m \in C(V) = \operatorname{Hom}(T(V), V)$, in other words, a collection of maps $m_k : V^k \to V$, satisfying, for any $v_1, \cdots, v_n \in V$, the relations

$$(1) \quad \sum_{\substack{k+l=n+1 \\ 0\leq i\leq n-k}} (-1)^{s_{i,k,n}} m_l(v_1, \cdots, v_i, m_k(v_{i+1}, \cdots, v_{i+k}), v_{i+k+1}, \cdots, v_n) = 0.$$

where $s_{i,k,n} = (v_1 + \cdots + v_i)k + i(k-1) + n - k$. In addition, we require that m_k have parity k, so that m_2 is an even map, and m_1 is an odd map. The map m_1 determines a differential on V, and is a derivation with respect to the product that m_2 determines on V. Also, the map m_2 is associative up to a homotopy determined by the element m_3. These statements follow from the relations given in equation (1) for $n = 1$, 2, and 3. The last fact accounts for the term **strongly homotopy associative algebras**, which is another name for A_∞ algebras.

If we let $\Pi W = V$, then there is a natural isomorphism η between $T(W)$ and $T(V)$, given by

$$\eta(w_1 \otimes \cdots \otimes w_n) = (-1)^{(n-1)v_1 + (n-2)v_2 + \cdots + (1)v_{n-1}} \pi w_1 \otimes \cdots \otimes \pi w_n. \tag{2}$$

Note that the restriction of η to W^k is a map $\eta_k : W^k \to V^k$, which has parity k, so this map is unusual because it is not homogeneous. The map η induces an isomorphism between $C(V)$ and $C(W)$, given by $m_k \mapsto d_k$, with $d_k = \eta^{-1} \circ m_k \circ \eta$. Now any element(cochain) $d \in C(W)$ extends to a $\mathbb{Z}_2$-graded coderivation on the coalgebra $T(W)$. In [13], it was shown that m determines an A_∞ structure on V precisely when $d = \eta^{-1} \circ m \circ \eta$ determines an odd codifferential on $T(W)$. (Actually this fact is well known, and really is the basis for the definition of A_∞ algebra in the first place.) We shall regard W, equipped with the structure given by d, as an A_∞ algebra. Now the condition that d determines a codifferential on $C(W)$ is given by

$$\sum_{\substack{k+l=n+1 \\ 0 \le i \le n-k}} (-1)^{(w_1 + \cdots + w_i)d_k} d_l(w_1, \cdots, w_i, d_k(w_{i+1}, \cdots, w_{i+k}), w_{i+k+1}, \cdots, w_n) = 0, \tag{3}$$

for any $w_1, \cdots, w_n \in W$. Because of the realization of elements in $C(W)$ as coderivations, there is a natural $\mathbb{Z}_2$-graded bracket on $C(W)$. The bracket originates with the associative product on $C(W)$ given by

$$\begin{aligned} d\delta(w_1, \cdots, w_n) = \\ \sum_{\substack{k+l=n+1 \\ 0 \le i \le n-k}} (-1)^{(w_1 + \cdots + w_i)\delta_k} d_l(w_1, \cdots, w_i, \delta_k(w_{i+1}, \cdots, w_{i+k}), w_{i+k+1}, \cdots, w_n), \end{aligned} \tag{4}$$

for d, $\delta \in C(W)$. Then $[d, \delta] = d\delta - (-1)^{d\delta} \delta d$. One can express this in the form

$$[d, \delta]_n = \sum_{k+l=n+1} [d_k, \delta_l], \tag{5}$$

where d_k represents the degree k part of d, etc. Since d is odd, the fact that it is a codifferential can be expressed in the simple form $[d, d] = 0$. The differential D determining the cohomology of the A_∞ algebra is given by $D(\delta) = [\delta, d]$, and the fact that $D^2 = 0$ follows from the ($\mathbb{Z}_2$-graded) Jacobi identity. The homology of $C(W)$, or the corresponding homology of $C(V)$, is called the Hochschild cohomology of the A_∞ algebra.

Now suppose that V is equipped with a graded symmetric inner product, that is a non-degenerate bilinear form $h : V^2 \to \mathbf{k}$, satisfying $h(v_1, v_2) = (-1)^{v_1 v_2} h(v_2, v_1)$.

(Non-degeneracy means that the induced map $h' : V \to V^*$, given by $h'(v_1)(v_2) = h(v_1, v_2)$ is an isomorphism.) Denote $h(v_1, v_2) = \langle v_1, v_2 \rangle$ for simplicity. The inner product induces an isomorphism between $C^n(V) = \mathrm{Hom}(V^n, V)$ and $C^{n+1}(V, \mathbf{k}) = \mathrm{Hom}(V^{n+1}, \mathbf{k})$, given by $\varphi \mapsto \tilde{\varphi}$, where

$$\tilde{\varphi}(v_1, \cdots, v_{n+1}) = \langle \varphi(v_1, \cdots, v_n), v_{n+1} \rangle . \tag{6}$$

$\varphi_n \in C^n(V)$ is said to be cyclic with respect to the inner product if

$$\langle \varphi_n(v_1, \cdots, v_n), v_{n+1} \rangle = (-1)^{n+v_1\mu} \langle v_1, \varphi_n(v_2, \cdots, v_{n+1}) \rangle . \tag{7}$$

This is equivalent to the condition that $\tilde{\varphi}_n$ is cyclic in the sense that

$$\tilde{\varphi}_n(v_1, \cdots, v_{n+1}) = (-1)^{n+v_{n+1}(v_1+\cdots+v_n)} \tilde{\varphi}_n(v_{n+1}, v_1, \cdots, v_n). \tag{8}$$

A cochain φ is said to be cyclic if φ_n is cyclic for all n. Note that $\tilde{\varphi}_n$ is cyclically symmetric for even n, but cyclically antisymmetric for odd n. This feature contributed to the difficulty in defining the homology cycle in [14]. The inner product is said to be invariant with respect to the A_∞ structure m if m is cyclic with respect to the inner product. The $\mathbb{Z}_2$-graded bracket on $C(W)$ induces a $\mathbb{Z}_2 \times \mathbb{Z}$-graded bracket $\{\cdot, \cdot\}$ on $C(V)$, and it was shown in [13] that the bracket of two cyclic cochains is again cyclic. For an invariant inner product, the cyclic elements form a subcomplex $CC(V)$ of $C(V)$, and the homology of this subcomplex is called the cyclic cohomology of the A_∞ algebra.

Let us recast the notion of cyclic homology in terms of the complex $C(W)$. We shall see that the definition becomes more natural. First let us consider the inner product k on W induced by the inner product h on V. This inner product is simply $h \circ \eta$, and it is easy to check that

$$k(w_2, w_1) = (-1)^{w_1 w_2 + 1} k(w_1, w_2), \tag{9}$$

so that k is graded antisymmetric.

Now one can define cyclicity in $C(W)$ in one of three manners. We could simply define ψ to be cyclic if $\psi = \eta^{-1} \circ \varphi \circ \eta$, for a cyclic cochain φ in $C(V)$. On the other hand, one could define ψ to be cyclic the corresponding element $\tilde{\psi} \in C(V, \mathbf{k})$ given by $\tilde{\varphi} = \tilde{\psi} \circ \eta^{-1}$ is cyclic. It turns out that these notions are equivalent, and in addition they are equivalent to the following definition of cyclicity. We say that $\psi_n \in C^n(W)$ is cyclic with respect to the inner product k if

$$\langle \psi(w_1, \cdots, w_n), w_{n+1} \rangle = (-1)^{w_1 \psi + 1} \langle w_1, \psi(w_2, \cdots, w_{n+1}) \rangle \tag{10}$$

This notion is also equivalent to $\tilde{\psi}$ being cyclic in the sense that

$$\tilde{\psi}(w_1, \cdots, w_{n+1}) = (-1)^{w_{n+1}(w_1+\cdots+w_n)} \tilde{\psi}(w_{n+1}, w_1, \cdots, w_n) \tag{11}$$

Note that $\tilde{\psi}$ is cyclically symmetric for all n. Next, we note that if δ and ψ are cyclic cochains in $C(W)$, then their bracket is cyclic, and in fact, we have the following theorem.

Theorem 1. *Suppose that W is a $\mathbb{Z}_2$-graded vector space with a graded antisymmetric inner product.*

a) If δ, $\psi \in C(W)$ are cyclic with respect to the inner product, then their bracket is also cyclic. Moreover, the following formula holds.

$$\widetilde{[\delta,\psi]}(v_1,\cdots,v_{n+1}) = \sum_{\substack{k+l=n+1\\0\le i\le n}} (-1)^{(w_1+\cdots+w_i)(w_{i+1}+\cdots+w_{n+1})}\tilde{\delta}_l(\psi_k(w_{i+1},\cdots,w_{i+k}),w_{i+k+1},\cdots,w_i). \tag{12}$$

Thus there is a bracket defined on the complex $CC(W,\boldsymbol{k})$ of cyclic elements in $C(W,\boldsymbol{k})$, by $[\tilde{\delta},\tilde{\psi}]=\widetilde{[\delta,\psi]}$. b) If d is a $\mathbb{Z}_2$-graded codifferential on $T(W)$, then there is a differential D in $CC(W,\boldsymbol{k})$, given by

$$D(\tilde{\psi})(w_1,\cdots,w_n) = \sum_{\substack{k+l=n+1\\0\le i\le n}} (-1)^{(w_1+\cdots+w_i)(w_{i+1}+\cdots+w_{n+1})}\tilde{\psi}_l(d_k(w_{i+1},\cdots,w_{i+k}),w_{i+k+1},\cdots,w_i). \tag{13}$$

c) If the inner product is invariant, then $D(\tilde{\psi})=[\tilde{\psi},\tilde{d}]$. Thus $CC(W,\boldsymbol{k})$ inherits the structure of a differential graded Lie algebra.

This result is really just a restatement of the corresponding result for cyclic cohomology formulated in terms of $C(V)$ (see citepen1). The form in which we will need to use this result is as follows. Suppose that $e_1,\cdots,e_n$ is a basis of W. (We assume always that V is finite dimensional.) The structure constants for the algebra are given by $d_k(e_{j_1},\cdots,e_{j_k})=d^a_{j_1,\cdots,j_k}e_a$. (We use the summation convention for repeated upper and lower indices.) Define the lower structure constants $d_{j_1,\cdots,j_{k+1}}$ of the A_∞ algebra by

$$d_{j_1,\cdots,j_{k+1}}=\tilde{d}_k(e_{j_1},\cdots,e_{j_{k+1}}). \tag{14}$$

If the inner product is denoted by $k_{\alpha\beta}=\langle e_\alpha,e_\beta\rangle$, then $k^{\alpha\beta}$ denotes the inverse matrix to k. Then the lower structure constants are given in terms of the usual ones by $d^a_{j_1,\cdots,j_k}e_a=h^{a,j_{k+1}}d_{j_1,\cdots,j_{k+1}}$. Then we can formulate the definition of an A_∞ algebra with an invariant inner product in terms of the structure constants by

$$\sum_{\substack{k+l=n+1\\0\le i\le n}} (-1)^{(e_{j_1}+\cdots+e_{j_i})(e_{j_{i+1}}+\cdots+e_{i_{n+1}})}d^a_{j_{i+1},\cdots,j_{i+l}}d_{a,j_{i+l+1},\cdots,j_i}=0. \tag{15}$$

This equation will play an important role in the construction of the cycle in the homology of the complex of metric ribbon graphs. Let us also summarize the additional information that we shall need. Although we did not state this before, we shall need to assume that the inner product is an even map. It is also antisymmetric, so that the tensor $K=k^{ab}e_a\otimes e_b$ is an even, antisymmetric tensor. The tensor $C_k=d_{j_1,\cdots,j_{k+1}}e^{j_1}\otimes\cdots\otimes e^{j_k}$ is an odd, cyclically symmetric tensor.

3. L_∞ ALGEBRAS

In this section we again assume that V is a finite dimensional $\mathbb{Z}_2$-graded vector space over a field $\mathbf{k}$of characteristic zero. The exterior coalgebra $\bigwedge V$ is defined by $\bigwedge V = \bigoplus_{n=1}^{\infty} \bigwedge^n V$, where $\bigwedge^n V$ is the n-th exterior power of V. An L_∞ structure on V is given by an element (cochain) $l \in thecomplex C(V) = \operatorname{Hom}(\bigwedge V, V)$, so that l is a collection of maps $l_k : \bigwedge V \to V$. We require that the relations

$$\sum_{\substack{k+l=n+1\\ \sigma\in\operatorname{Sh}(l,n-l)}} (-1)^{\sigma}\epsilon(\sigma)(-1)^{(k-l)l} l_l(l_k(v_{\sigma(1)},\cdots,v_{\sigma(k)}),v_{\sigma(k+1)},\cdots,v_{\sigma(n)} = 0. \tag{16}$$

where $\operatorname{Sh}(l, n-l)$ is the set of unshuffles of type $(l, n-l)$, in other words, permutations σ satisfying $\sigma(i) \le \sigma(i+1)$ unless $i = l$, $(-1)^\sigma$ is the sign of the permutation, and $\epsilon(\sigma)$ is a sign which depends on both σ and $v_1, \cdots, v_n$. For a more detailed explanation of these signs, see [13]. In addition, the maps l_k are of parity k. In complete parallel to the A_∞ picture, we have l_1 is an odd differential on V, acting as a derivation on the multiplication l_2, which satisfies the Jacobi identity up to a homotopy determined by l_3. L_∞ algebras are also called **strongly homotopy Lie algebras**. Lie algebras and differential graded Lie algebras are special cases of L_∞ algebras.

Let $\Pi W = V$ be the parity reversion of V. There is a natural isomorphism $\eta : \bigodot W \to \bigwedge V$, where $\bigodot W$ is the symmetric coalgebra of W, which is defined by

$$\eta(w_1 \odot \cdots \odot w_n) = (-1)^{(n-1)w_1+\cdots+w_{n-1}} \pi w_1 \wedge \cdots \wedge w_n. \tag{17}$$

If we denote $C(W) = \operatorname{Hom}(\bigodot W, W)$, then the isomorphism η induces an isomorphism between $C(V)$ and $C(W)$ given by $\varphi \mapsto \eta^{-1} \circ \varphi \circ \eta$. It is well known that l determines an A_∞ structure precisely when its image d under this isomorphism determines a $\mathbb{Z}_2$-graded odd codifferential on $\bigodot W$. We shall regard W equipped with d as an L_∞ algebra. The statement that d determines a codifferential is equivalent to

$$\sum_{\substack{k+l=n+1\\ \sigma\in\operatorname{Sh}(l,n-l)}} (-1)^{\sigma}\epsilon(\sigma) d_l(d_k(v_{\sigma(1)},\cdots,v_{\sigma(k)}),v_{\sigma(k+1)},\cdots,v_{\sigma(n)} = 0. \tag{18}$$

Elements of $C(W)$ extend uniquely to coderivations on $\bigodot W$, so $C(W)$ has a $\mathbb{Z}_2$-graded Lie bracket, and the codifferential d determines a differential D on $C(W)$ by $D(\psi) = [\psi, d]$. This differential determines the homology of the L_∞ algebra. Cocycles of a Lie algebra give rise to infinitesimal deformations of the Lie algebra into an L_∞ algebra.

Now suppose V has a $\mathbb{Z}_2$-graded symmetric inner product h, so that W is equipped with a $\mathbb{Z}_2$-graded antisymmetric inner product k. An element $\varphi \in \operatorname{Hom}(\bigwedge^n V, V)$ is cyclic if it satisfies equation (7). The map φ is cyclic if and only if the map $\tilde{\varphi}$, given by equation (6) is antisymmetric, in other words, $\tilde{\varphi} \in C^{n+1}(V, \mathbf{k}) = \operatorname{Hom}(\bigwedge^{n+1} V, \mathbf{k})$. Denote the set of cyclic elements by $CC(V)$. The Lie bracket on $C(W)$ induces a bracket on $C(V)$, and the bracket of two cyclic elements in $C(V)$ is again cyclic, so $CC(V)$ is a Lie subalgebra of $C(V)$. If the inner

product is invariant, then $CC(V)$ is a subcomplex of $C(V)$. The cyclic cohomology of the L_∞ is the homology of this subcomplex.

One also can define cyclic cohomology using the complex $C(W)$, and the antisymmetric inner product k. The definition of cyclicity of an element $\psi \in C(W)$ is again given by equation (10). The notion of cyclicity is equivalent to the map $\tilde{\psi}$, given by $\tilde{\psi}(w_1, \cdots, w_{n+1}) = \langle \psi(w_1, \cdots, w_n), w_{n+1} \rangle$, being graded symmetric. The bracket of cyclic elements in $C(W)$ is again cyclic, and so we obtain the following theorem.

Theorem 2. *Suppose that W is a $\mathbb{Z}_2$-graded vector space with a graded antisymmetric inner product. Let $C(W) = Hom(\bigodot W, W)$ be equipped with the bracket induced by the identification of elements in $C(W)$ with coderivations of $\bigodot W$.*

a) If φ, $\psi \in C(W)$ are cyclic with respect to the inner product, then their bracket is also cyclic. Moreover, the following formula holds.

$$\widetilde{[\varphi, \psi]}(v_1, \cdots, v_{n+1}) = \sum_{\substack{k+l=n+1 \\ \sigma \in \mathrm{Sh}(l, n+1-l)}} (-1)^\sigma \epsilon(\sigma) \tilde{\varphi}_l(\psi_k(w_{\sigma(1)}, \cdots, w_{\sigma(k)}), w_{\sigma(k+1)}, \cdots, w_{\sigma(n+1)}). \tag{19}$$

Thus there is a bracket defined on the complex $CC(W, \boldsymbol{k})$ of cyclic elements in $C(W, \boldsymbol{k})$, by $[\tilde{\varphi}, \tilde{\psi}] = \widetilde{[\varphi, \psi]}$.

b) If d is a $\mathbb{Z}_2$-graded codifferential on $T(W)$, then there is a differential D in $CC(W, \boldsymbol{k})$, given by

$$D(\tilde{\psi})(w_1, \cdots, w_{n+1}) = \sum_{\substack{k+l=n+1 \\ \sigma \in \mathrm{Sh}(l, n+1-l)}} (-1)^\sigma \epsilon(\sigma) \tilde{\varphi}_l(d_k(w_{\sigma(1)}, \cdots, w_{\sigma(k)}), w_{\sigma(k+1)}, \cdots, w_{\sigma(n+1)}) \tag{20}$$

c) If the inner product is invariant, then $D(\tilde{\psi}) = [\tilde{\psi}, \tilde{d}]$. Thus $CC(W, \boldsymbol{k})$ inherits the structure of a differential graded Lie algebra.

The form in which we need these results is as follows. Let $e_1, \cdots, e_m$ be a basis of W, and define the lower structure constants of the L_∞ algebra $d_{j_1, \cdots, j_k}$ by equation (14). Then we can reformulate the definition of an L_∞ algebra with an antisymmetric invariant inner product in terms of the structure constants by

$$\sum_{\substack{k+l=n+1 \\ \sigma \in \mathrm{Sh}(l, n-l)}} \epsilon(\sigma) d^a_{j_{\sigma(1)}, \cdots, j_{\sigma(l)}} d_{a, j_{\sigma(l+1)}, \cdots, j_{\sigma(n)}} = 0. \tag{21}$$

4. Definition of the Graph Complexes

A graph is a 1 dimensional CW complex, in other words, it consists of vertices and edges. We will consider here only graphs where each vertex is at least trivalent, meaning that the number of edges incident to the vertex is at least 3. The graph is called a ribbon graph if in addition there is a fixed cyclic order of the edges at each vertex. A metric on the graph is an assignment of a positive number to each edge of the graph. The set σ_Γ of all metrics on a graph gives a cell, identifiable with R^k_+ where $k = e(\Gamma)$ is the number of edges in the graph. Intuitively, when

the length of an edge of a graph tends to zero, the graph degenerates to a new graph Γ', which determines a cell on the boundary of the cell σ_Γ. We shall give a more precise definition of the boundary operator shortly. Let us point out that neither the space of metric ribbon graphs, nor that of metric ordinary graphs is a cell complex, because the closure of a cell is not compact. (Boundary cells exist only on the sides corresponding to setting a length equal to zero, not on the infinite sides.)

If we consider the space of ordinary graphs, then the Euler Characteristic is preserved when contracting edges, so there are different graph complexes corresponding to the different Euler characteristics. For the space of metric ribbon graphs, each metric graph corresponds to a Riemann surface with a fixed genus g and number of marked points n, depending only on the graph (see [4, 16]). The boundary operator also preserves the genus and number of marked points. Thus for ribbon graphs, we have different graph complexes for each genus and number of marked points.

To define a boundary operator, we shall need a notion of the orientation of the cells in the graph complex. Let us first proceed with the case of ribbon graphs. An orientation of the complex of metric ribbon graphs is determined by a ordering of the edges in the graph, and a ordering of the holes or circuits in the graph (see [6, 14]). Using this notion of orientation, a cycle in the complex of metric ribbon graphs associated to an A_∞ algebra with a symmetric invariant inner product was constructed in [14]. This construction was dependent on the non trivial fact that the complex of metric ribbon graphs is orientable. Here we propose an equivalent definition of the orientation of a cell corresponding to a graph Γ. The orientation of the cell is given by choosing a ordering of the vertices and assigning an arrow to each edge (in other words, choosing an orientation of the edge). The fact that this orientation is equivalent to the previous one is not obvious, so here we also are concealing some non trivial relations. For the case of metric ordinary graphs, in [7] an orientation was defined to be an ordering of the edges in the graph, as well as an orientation of the first homology group. It is not difficult in this case to see that this notion of orientation is equivalent to the definition of an orientation as an ordering of the vertices and an assignment of an orientation to each edge of the graph. Thus our definition of orientation is the same for ribbon graph and ordinary graph complexes.

Next, we would like to define the boundary operator. Let us make the abuse of notation and identify the cell σ_Γ with the graph Γ. Two oriented graphs are considered to be the same graph if there is an isomorphism between the graphs which induces the same orientation. If we contract out an edge in an oriented graph, then there is a natural orientation induced in the contracted graph from an orientation in the original graph. To see this, suppose that the graph Γ_{or} has edges labeled from 1 to n, and vertices labeled from 1 to m. When an edge is contracted out, a vertex also is contracted. An orientation only determines the labels on the edges and vertices up to an even permutation, so it is always possible to assume that the labeling has been chosen so that the edge to be contracted has a vertex labeled as m, and that the arrow points towards this vertex. Then, when contracting, the vertex with label m disappears. However, the process is reversible, since when inserting an edge, one has a choice of which of the two vertices that occur will have the new label, requiring that the arrow in the inserted edge point to the new vertex fixes the orientation uniquely. Notice that the same construction

applies to ordinary and to ribbon graphs. The only special feature for ribbon graphs is that when contracting edges, in combining the vertices, there is an induced cyclic order on the vertex, and similarly, when inserting an edge, there is a natural way to induce a cyclic order at the new vertices from the cyclic order on the old vertex.

The graph complex is generated by the classes of oriented graphs, with relations given by identifying an oriented graph with the negative of the same graph with opposite orientation. In our construction, we are taking coefficients in the field $\mathbf{k}$, which has characteristic zero, so that if an oriented graph is equivalent to its opposite orientation, then it is zero in the graph complex. But it is also interesting to consider the case of coefficients in $\mathbb{Z}_2$, in which case orientation plays no role, so that one obtains a different homology theory. A graph with a loop is equivalent to its opposite orientation. To see this, note that an equivalence of oriented graphs is given by an equivalence of graphs, which assigns a vertex to each vertex, and thus induces an ordering of the vertices. Normally, one can assign an arrow to an arrow, by requiring that the induced arrow on an edge point from the induced starting vertex to the induced ending vertex, but this procedure is ambiguous when the starting and ending vertex is the same. Thus either orientation can occur as the image of the oriented graph.

Let us introduce a boundary operator in the following manner. If Γ_{or} and ${\Gamma'}_{or}$ are oriented graphs, then we need to assign an incidence number $[{\Gamma'}_{or}, \Gamma_{or}]$, which counts the number of times that ${\Gamma'}_{or}$ occurs in the boundary of Γ_{or}. Let G be the set of all distinct equivalence classes in the complex. Graphs with opposite orientation determine the same equivalence class in this context. In the case of ribbon graphs, we can choose G to be the set of all equivalences of some fixed genus g and number of marked points n, while for the ordinary graph complex, we can choose G to be the set of all equivalence classes of the same Euler characteristic. A chain in the complex can be represented as a sum of the form $\sum_{\Gamma_{or}\in G} a_{\Gamma_{or}}\Gamma_{or}$, Then the boundary operator is given by

$$\partial(\Gamma_{or}) = \sum_{\Gamma_{or}\in G} [{\Gamma'}_{or}, \Gamma_{or}]{\Gamma'}_{or} \tag{22}$$

To determine $[{\Gamma'}_{or}, \Gamma_{or}]$, we consider the result of contracting an edge in Γ_{or}. An edge is contractable if it is attached to two distinct vertices. When an edge is contracted, the graph ${\Gamma'}_{or}$ or its opposite may occur, and if so, one counts either a plus one or a minus one, depending on which occurs. The sum of the numbers resulting from contracting the various edges in Γ_{or} is the desired incidence number.

Figures 1 and 2 illustrate why the square of the boundary operator is zero. When the boundary operator is applied twice, that is the same as contracting two edges in the graph. These edges may be contracted in two orders, and the resulting graphs will be the same. Thus we need only show that the incidence numbers associated to these two different orders are opposite, to show that the square of the boundary operator is zero. In figure 1 the orientations in A and B are the same. Recall that to contract an edge, one should place the highest number of a vertex at one of the edges, and the arrow on the edge should face that vertex. Thus in A, we can contract the edge from vertex 1 to 3, and then the arrows will be in accord so that the remaining edge can be contracted. However, in B, we can contract the edge from 3 to 2, but then the remaining arrow is pointing in the wrong direction. This

shows that the consequence of contracting the two edges in the opposite order is that the incidence numbers will cancel.

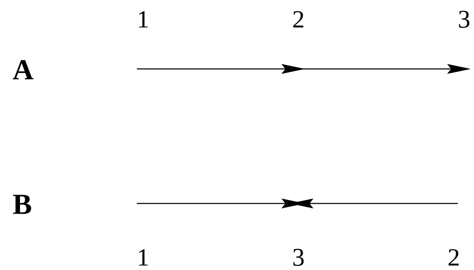

FIGURE 1. Contracting adjacent edges

In figure 1, the two edges were assumed to be adjacent. If the edges are not adjacent, one can arrange the labels as in figure 2 below. The orientations in A and B below are the same. In A, one can contract first the edge with vertices labeled 1 and 4, and then the edge with vertices labeled 3 and 2. The same procedure is applied to B. The resulting graphs have opposite orientation, since the position of the vertices 1 and 2, which remain after the contraction, is reversed.

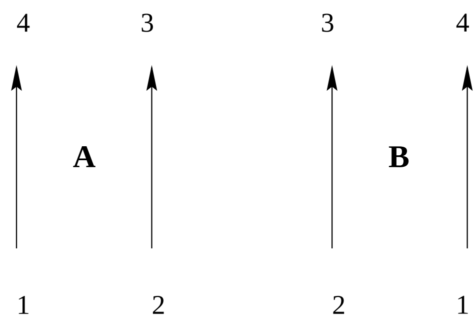

FIGURE 2. Contracting non adjacent edges

5. THE CYCLE GIVEN BY AN A_∞ ALGEBRA

Let W be finite dimensional vector space with basis e_i, equipped with an A_∞ algebra structure given by a codifferential d on $T(W)$. Suppose that k is a graded antisymmetric inner product on W which is invariant with respect to the A_∞ structure, and that the tensors $K = k^{ab} e_a \otimes e_b$ and $C_n = d_{j_1,\cdots,j_n} e^{j_1} \otimes \cdots \otimes e^{j_n}$ are defined as in section 2. Recall that K is antisymmetric and even, and that C_n is odd and cyclically symmetric. We want to construct a cochain in the graph complex, which is a function Z on the set of oriented graphs such that $Z(\Gamma_{or})$ has opposite sign for graphs of opposite orientation. To define Z, let Γ_{or} be an oriented graph. and to each edge in the graph associate two indices, one for each incident vertex. Intuitively, if the edge has labels i and j, then associate the tensor h^{ij}, and if the labels on a vertex with n incident edges are $j_1, \cdots, j_n$ in cyclic order, then associate

the tensor $d_{j_1,\cdots,j_n}$ to the vertex. Multiply the tensors for the vertices in the same order as the ordering of the vertices, and then multiply the tensors corresponding to the edges in any order. Then one obtains a tensor in which every upper indice matches precisely one lower index, so summing over repeated indices yields a number $\mathrm{Z}(\Gamma_{or})$. Figure 3 below illustrates the idea. The vertices are labeled 1 and 2. The cyclic order at each vertex is assumed to be counterclockwise. The portion of the product corresponding to the figure is $d_{glmih}d_{nkj}k^{bh}k^{ic}k^{mn}k^{lf}k^{ag}k^{jd}k^{ek}$.

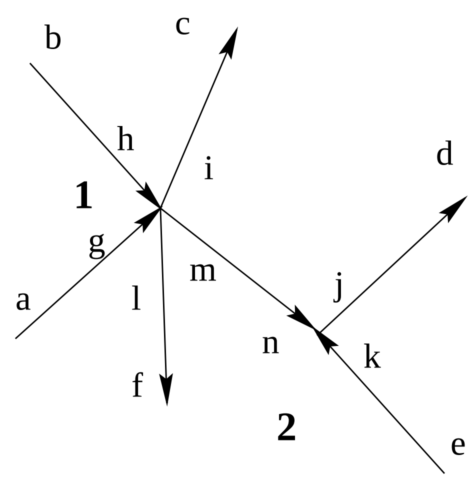

Figure 3. Constructing the cycle

More precisely, one should consider the product of tensors C_n, in the order dictated by the order of the vertices, and the tensors K, one for each edge. This will yield an element of $(W^*)^{2e} \otimes W^{2e}$. Then one uses the graph to determine a graded contraction of this tensor, obtaining an element of $\mathbf{k}$. In the informal construction, one must take into account a sign that arises from the graded contraction as well. The sign corresponding to the above portion arises as a consequence of the exchange rule in computing the contraction. For example, if one contracts the tensor $e^a \otimes e^b \otimes e_a \otimes e_b$ according to the order indicated, then one obtains the sign $(-1)^{e^b e_a}$, but the contraction of the tensor $e^a \otimes e^b \otimes e_b \otimes e_a$ yields the sign 1.

We need to show that the definition of the number $Z(\Gamma_{or})$ does not depend on any of the choices that we made in writing down the tensor. First of all, since the tensors C_n are odd, permuting their order will introduce a sign corresponding exactly to the sign of the permutation, so this is precisely what we need to have the sign depend properly on the orientation. Secondly, C_n is cyclically (graded) symmetric, so that the starting point we used to write the tensor down makes no difference. Thirdly, the order in which the tensors K are written down makes no difference, since they are even tensors. Finally, the order in which the indices are written down must conform to the order in which the arrow appears, and the antisymmetry guarantees that the reversal of the arrow will reverse the sign in the

contraction. Thus we have shown that

$$Z = \sum_{\Gamma_{or} \in G} Z(\Gamma_{or})\Gamma_{or} \tag{23}$$

is a well defined chain in the complex.

Next, we wish to show that Z is a cycle. It is sufficient to consider a graph Γ'_{or}, and consider a single vertex in Γ'_{or}, and the graphs which are obtained by inserting an edge in the vertex. We show that the sum of the contributions of each of these graphs to the boundary of Z is zero. Let us consider how an edge might be inserted in a vertex with $n+1$ incident edges to obtain two new vertices with $k+1$ and $l+1$ incident edges, where $k+l = n+1$, as is illustrated in the figure below.

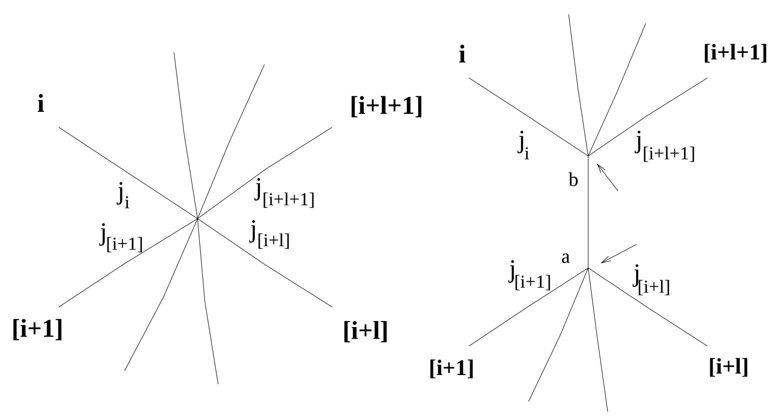

Figure 4. Splitting a vertex

When inserting a new edge, one specifies two incident edges, j_i and j_{i+l} in the diagram. If the cyclic order at the vertex is counterclockwise, then the cyclic order at the two new vertices is counterclockwise. Denote the expanded graph by Γ_{or}. Let us suppose that the vertex with new incident edge labeled b is the one which is given the new highest label. Then the arrow runs from a to b, and the contribution to the function $Z(\Gamma_{or})$ from these two vertices and the new edge is $d_{j_{i+1},\cdots,j_{i+l},a} d_{b,j_{i+l+1},\cdots,j_i} h^{ab}$. Since $d^b_{j_{i+1},\cdots,j_{i+1}} = d_{j_{i+1},\cdots,j_{i+l},a} h^{ab}$, and the sign corresponding to the graded contraction is $(-1)^{(e_{j_1}+\cdots+e_{j_i})(e_{j_{i+1}}+\cdots+e_j n+1)}$. we see that this number is just the i-th term in the expression given in equation 15. Since the sum of all the terms is zero. we conclude that the contribution of all the graphs which arise from the insertion of an edge in Γ'_{or} is zero. Thus Z is a cycle. In the proof, we are tacitly assuming that the d_1 term in the A_∞ algebra vanishes, but the proof can be extended to show that the chain Z is still a cycle when the d_1 term does not vanish, see [12].

In [14], we showed that if A was an associative algebra with product m, and φ was a k-cocycle, then one obtains a cycle in the homology of the graph complex by considering the infinitesimal deformation $m_t = m + t\varphi$ of the associative algebra into an A_∞ algebra. The cycle so constructed depends only on graphs which have trivalent vertices, with the exception of one vertex which is k-valent. One can prove

this directly as well, using the corresponding formula for the cyclic coboundary operator on the parity reversion. Albert Schwarz conjectured that the cycle obtained in this manner was independent of the algebra in the sense that one always obtains a multiple of the same cycle. As it turns out, this conjecture follows from a theorem of Penner in [15], because there is only one cycle of this type.

6. The cycle given by an L_∞ algebra

The cycle in the graph complex of metric ordinary graphs is constructed in the same manner as the cycle in the complex of ribbon graphs, so I shall just describe the proof that it is a cycle, which is a little different. For ordinary graphs, there is no cyclic order at each vertex, so the number of ways in which an edge can be inserted at a vertex is much larger. For example, if a vertex has four incident edges, labeled a, b, c and d, then if the graph is a ribbon graph, and the cyclic order is a, b, c, d, one can insert an edge so that a and b are paired with one of the vertices, and c and d with the other. Alternatively, one can insert an edge so that b and c pair with one of the vertices, and d and a with the other. No other possibility occurs. However, for the ordinary graph complex, we cannot distinguish any order, so there is in addition, the possibility that a and c pair off with a new vertex, and b and d with the other. The combinations that can occur when decomposing a vertex with $n+1$ incident edges into two vertices with $l+1$ and $k+1$ edges respectively are given by the shuffles of type (l, k). Let ${\Gamma'}_{or}$ be a graph and consider a vertex with $n+1$ incident edges labeled $j_1, \cdots, j_{n+1}$. If σ is such a shuffle, then let Γ_{or} be the graph which results from inserting an edge so that the edges $\sigma(1), \cdots, \sigma(l)$ are associated to one of the new vertices, and the other edges with the other vertex. Then the contribution to $Z(\Gamma_{or})$ from the new vertices and the edge is $d_{j_{\sigma(1)}, \cdots, j_{\sigma(l)}, a} d_{b, j_{\sigma(l+1)}, \cdots, j_{\sigma(n+1)}}$. The sign corresponding to the graded contraction turns out to be just $\epsilon(\sigma)$. Then it is clear from equation (21) that the sum of the terms corresponding to the various ways to insert the new edge cancel in their contribution to the boundary of Z.

In [13], it was shown that if L is a Lie algebra with bracket l and invariant inner product, then a cyclic cocycle φ determines an infinitesimal deformation of the Lie algebra into an L_∞ algebra with invariant inner product. Suppose that the cocycle is of exterior degree n, in other words, $\varphi \in C^n(V)$. Then we can associate to this cocycle a cycle in the homology of the ordinary graph complex which depends only on the graphs with all vertices trivalent, except for one vertex with $n+1$ edges. I do not know if the cycle depends on the algebra.

7. Conclusion

This paper has shown how to construct cycles in the homology of graph complexes from A_∞ and L_∞ algebras with invariant inner products. A corollary of this construction is the existence of cycles in the graph complexes associated to cyclic cocycles of associative and Lie algebras with invariant inner products. These cycles exist because of the connection between the cyclic cohomology of these algebras and infinitesimal deformations of the algebras into infinity algebras. One can also consider higher order deformations, leading to the construction of some other cycles in the homology of the graph complexes.

There is a general theory of homotopy algebras, which gives A_∞ and L_∞ algebras as special cases. This theory was introduced in [3] In addition, there is a unified theory, called operad cohomology, which was introduced by T. Fox and M. Markl in [8]. The graph complexes are closely related to certain operad structures. There must be some relation between the results here and the operads which are used to define these homotopy algebras. Probably a simple proof of these results can be given, using operad theory.

One can generalize the graph complexes to allow vertices with only two edges. In this more general complex, it is immediate that one can define cycles using L_∞ and A_∞ terms which do not have trivial d_1 term. However, it was shown in [12], and is not difficult to see, that the generalized graph complex is a direct sum of the ordinary graph complex and another subcomplex. As a consequence, it follows that L_∞ and A_∞ algebras with non trivial d_1 term give rise to cycles in the homology of the graph complex. One can also take the homology of an infinity algebra with respect to the d_1 term, and this homology inherits the structure of an infinity algebra with a trivial d_1 term. The cycle given by this homology algebra probably coincides with the cycle given by the original algebra, but I don't know of a proof of this result.

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University of California, Davis, CA 95616
E-mail address: michae@math.ucdavis.edu